JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 89, 530–557 (1982)

On the General Linear Coupled System for Diffusion in Media with Two Diffusivities

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Submitted by Alex McNabb

For the general linear coupled system of partial differential equations arising in the theory of diffusion in media with double diffusivity, simple uniqueness criteria, and a method of solution of boundary value problems are established. The equations studied retain the so-called *cross terms* which have been neglected in all previous investigations. Moreover, these equations arise as generalizations of a number of existing theories; for example, heat flow in heterogeneous multicomponent systems, flow of water in fissured rocks and a model of an arms race. The simple inequalities obtained on the various constants of the theory which guarantee uniqueness of solutions and existence of source solutions might serve as guidelines in an experimental determination of these constants. The solution procedure involves solving two boundary value problems for the classical diffusion equation and the formulae given mean that closed form expressions can be deduced for a number of commonly occurring boundary value problems. The paper emphasizes the general equations without special reference to particular physical applications or boundary value problems.

1. INTRODUCTION

In this paper we consider the coupled system of differential equations

$$\begin{aligned} \partial u_1/\partial t &= D_1 \nabla^2 u_1 + E_1 \nabla^2 u_2 - A_1 u_1 + B_1 u_2, \\ \partial u_2/\partial t &= D_2 \nabla^2 u_2 + E_2 \nabla^2 u_1 - A_2 u_2 + B_2 u_1, \end{aligned} \tag{1.1}$$

for $u_1(\mathbf{x}, t)$ and $u_2(\mathbf{x}, t)$. We assume that A_i, B_i, D_i, and E_i $(i = 1, 2)$ are all nonnegative constants and that we have labelled u_1 and u_2 such that $D_1 > D_2$. Equations (1.1) arise in a number of physical situations. For example, with $A_1 = B_2$ and $A_2 = B_1$ (1.1) has been deduced by Aifantis [1, 2] and Hill [8] by two distinct approaches for the phenomenon of diffusion in media with double diffusivity. In this context $u_1(\mathbf{x}, t)$ and $u_2(\mathbf{x}, t)$ denote concentrations. Aifantis [1, 2] derives (1.1) from a continuum approach while Hill [8] deduces these equations from a random walk model. In Aifantis and Hill [4] and Hill and Aifantis [9] uniqueness, maximum prin-

0022-247X/82/100530–28$02.00/0

ciples and a solution procedure are established for the system (1.1) with $A_1 = B_2$, $A_2 = B_1$, and E_1 and E_2 zero. The purpose of this paper is to extend the uniqueness results of [4] and the solution procedure of [9] to (1.1) with specifically E_1 and E_2 nonzero. In all previous physical theories involving equations of the form (1.1) it has been tacitly assumed that the E_1 and E_2 terms provide a negligible contribution. With E_1 and E_2 nonzero many of the results for (1.1) are considerably more complicated, especially formulae for solutions. Considering the numerous applications of the system, however, these results would appear to be worthwhile.

We can eliminate either u_1 or u_2 from (1.1) by equating to zero the determinant of the matrix with (1.1) written as

$$\begin{bmatrix} \dfrac{\partial}{\partial t} - D_1 \nabla^2 + A_1 & -E_1 \nabla^2 - B_1 \\ -E_2 \nabla^2 - B_2 & \dfrac{\partial}{\partial t} - D_2 \nabla^2 + A_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.2}$$

We find that both u_1 and u_2 satisfy the fourth order equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + (A_1 + A_2) \frac{\partial u}{\partial t} + (A_1 A_2 - B_1 B_2)\, u \\ &\quad = (A_1 D_2 + A_2 D_1 + B_1 E_2 + B_2 E_1) \nabla^2 u + (D_1 + D_2) \nabla^2 \frac{\partial u}{\partial t} \\ &\quad - (D_1 D_2 - E_1 E_2) \nabla^4 u. \end{aligned} \tag{1.3}$$

Although our principal concern is the coupled system (1.1) it is worthwhile bearing in mind the related equation (1.3), especially the particular combinations of constants occurring in this equation. For example, if we define D^* by

$$D^* = (A_1 D_2 + A_2 D_1 + B_1 E_2 + B_2 E_1)/(A_1 + A_2), \tag{1.4}$$

then in the context of diffusion in media with double diffusivity, namely, $A_1 = B_2$ and $A_2 = B_1$, we can identify D^* as the large time diffusivity in the sense that the asymptotic form of the total concentration $u = u_1 + u_2$ for the source solutions satisfies the classical diffusion equation as t tends to infinity, namely,

$$\partial u/\partial t = D^* \nabla^2 u. \tag{1.5}$$

In the following section we indicate the various areas of application in which (1.1) arises. In Section 3 we list the major results of the paper. In

Section 4 we show that sufficient conditions to deduce simple uniqueness results for (1.1) are

$$A_1, A_2, B_1, B_2 \geqslant 0, \qquad A_1A_2 - B_1B_2 \geqslant 0,$$
$$D_1, D_2, E_1, E_2 \geqslant 0, \qquad D_1D_2 - E_1E_2 \geqslant 0, \tag{1.6}$$

together with (4.13). We observe in Section 4 that inequalities (1.6) also guaranteee simple uniqueness properties for the fourth order equation (1.3). In Section 5 we deduce expressions for the basic source solutions of (1.1). These are solutions of (1.1) which vanish at infinity for all times while initially they are determined by

$$u_1(\mathbf{x}, 0) = \rho_1 \delta(\mathbf{x}), \qquad u_2(\mathbf{x}, 0) = \rho_2 \delta(\mathbf{x}), \tag{1.7}$$

where ρ_1 and ρ_2 are constants specifying the strength of the source and $\delta(\mathbf{x})$ denotes the usual Dirac delta function. We find that the constraints (1.6) are sufficient to deduce formal expressions for the source solutions provided we consider $(1.6)_3$ and $(1.6)_4$ as strict inequalities and that we supplement (1.6) with the inequality

$$(B_1E_2 + B_2E_1) < \tfrac{1}{2}(A_1 - A_2)(D_1 - D_2). \tag{1.8}$$

This latter inequality guarantees exponential decay of the source solutions with time. We remark that with the convention $D_1 > D_2$, (1.6) and (1.8) imply that $A_1 > A_2$. Moreover, from (1.4) and (1.8) we can deduce the interesting inequality

$$D^* < \tfrac{1}{2}(D_1 + D_2). \tag{1.9}$$

In Section 6 we deduce asymptotic expressions for the source solutions.

In Section 7 we make use of the expressions obtained for the source solutions to briefly outline the procedure deducing solutions of (1.1) in terms of two arbitrary solutions $\{h_1(\mathbf{x}, t), h_2(\mathbf{x}, t)\}$ of the classical diffusion equation with unit diffusivity, namely

$$\partial h/\partial t = \nabla^2 h. \tag{1.10}$$

These general expressions for solutions of (1.1) mean that solutions for initial value problems with zero boundary data can be deduced immediately from solutions of two initial value problems for (1.10) with the same zero boundary data. In order to utilise these general expressions for initial value problems with nonzero boundary data we need to invert these relations to express $\{h_1(\mathbf{x}, t), h_2(\mathbf{x}, t)\}$ as integrals involving $\{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\}$. This is done in Section 8. In Section 9 we obtain explicit results for commoinly occurring boundary data imposed on (1.1). In order to keep the lengthy calculations involved to a minimum we shall wherever possible refer the reader to Lee [10], where a more detailed explanation with examples can be found.

2. Applications

The purpose of this section is to indicate some of the applications in which the coupled system (1.1) might arise. We emphasize that those considered do not exhaust the areas of applicability of (1.1). Roughly speaking, equations such as (1.1) arise from the usual conservation laws (for example, conservation of mass) together with a generalization of Fick, Fourier, or Darcy type laws. For example, for the phenomenon of diffusion in a medium with two families of diffusion paths we identify two concentrations $u_1(\mathbf{x}, t)$ and $u_2(\mathbf{x}, t)$ for each of which conservation of mass yields

$$(\partial u_1/\partial t) + \nabla \cdot \mathbf{j}_1 = q, \qquad (\partial u_2/\partial t) + \nabla \cdot \mathbf{j}_2 = -q, \tag{2.1}$$

where $\mathbf{j}_1$ and $\mathbf{j}_2$ are the flux vectors for which we postulate the following generalizations of Fick's law, namely,

$$\mathbf{j}_1 = -D_1 \nabla u_1 - E_1 \nabla u_2, \qquad \mathbf{j}_2 = -D_2 \nabla u_2 - E_2 \nabla u_1, \tag{2.2}$$

where D_1, D_2, E_1, and E_2 are assumed to be nonnegative constants. Further, q in (2.1) represents the mass transfer from one path to the other, and following Aifantis [2], we propose the following form

$$q = -A_1 u_1 + A_2 u_2, \tag{2.3}$$

where A_1 and A_2 are nonnegative constants. Clearly, from (2.1) to (2.3) we can, in a straightforward manner, deduce (1.1). This is essentially the derivation given by Aifantis [1, 2]. A more concise derivation of (1.1) with $A_1 = B_2$, $A_2 = B_1$ for the same phenomenon is given by Hill [8] based on a random walk model. Hill [8] considers the random walk of a single particle on the real line which is assumed to simultaneously consist of two distinct paths. At intervals of time Δt, the particle can make a step Δx, to the right or left or remain in position; either along the same path or on the other path, moving simultaneously. From this formulation of the continuous random walk Hill [8] obtains for the free particle

$$\Delta t \frac{\partial f}{\partial t} = (\Delta x)^2 \left(p_{11} \frac{\partial^2 f}{\partial x^2} + p_{21} \frac{\partial^2 g}{\partial x^2} \right) - (r_{12} + 2p_{12}) f + (r_{21} + 2p_{21}) g,$$

$$\Delta t \frac{\partial g}{\partial t} = (\Delta x)^2 \left(p_{22} \frac{\partial^2 g}{\partial x^2} + p_{12} \frac{\partial^2 f}{\partial x^2} \right) - (r_{21} + 2p_{21}) g + (r_{12} + 2p_{12}) f, \tag{2.4}$$

where $f(x, t)$ and $g(x, t)$ denote probability density functions for the position of the particle at time t and $p_{ij}(i, j=1, 2)$, are the probabilities that the particle moves either to the left or to the right (assumed equal) from path i into path j, $r_{ij}(i, j=1, 2)$, is the probability that the particle stays at the same position but moves from path i into path j. We note that in order to include applications other than diffusion in media with two diffusivities we shall assume throughout that the constants A_1 and A_2 are not necessarily equal to B_2 and B_1, respectively.

Phenomena giving rise to special cases of (1.1) are examined by Rubinstein [13] and Barenblatt *et al.* [5]. Rubenstein [13] considers heat conduction in a heterogeneous multicomponent system where the heat transfer between components is assumed to obey Henry's law and further that Fourier's law of heat conduction is assumed within each component. For a two component system, Rubenstein [13] defines Δw as an elementary region of (x, y, z) space and takes v_i^* as the average temperature of the elements of component i contained in Δw. Distributions of the elements of the medium are considered and v_i is defined as the expectation of v_i^*. With a_i^2 being the coefficient of temperature conduction of component i, Rubenstein [13] obtains the coupled system

$$\frac{1}{a_1^2}\frac{\partial v_1}{\partial t}=\nabla^2 v_1+\alpha_1(v_2-v_1), \qquad \frac{1}{a_2^2}\frac{\partial v_2}{\partial t}=\nabla^2 v_2+\alpha_2(v_1-v_2). \tag{2.5}$$

In Barenblatt *et al.* [5] the seepage of homogeneous liquids in fissured rocks is considered. The rocks consist of permeable blocks separated by a system of fissures. Darcy's law is assumed along the fissures. In the vicinity of a given point two liquid pressures are introduced; p_1 the average pressure of the liquid in the fissures and p_2 the average pressure of the liquid in the pores of the permeable blocks. The porosity of the system of fissures is given by k_1', and k_2' is the porosity of the system of pores. The viscosity of the liquid in both the pores and fissures is given by μ. With α, β_1, and β_2 being positive constants the following system is obtained:

$$\beta_1\frac{\partial p_1}{\partial t}=\frac{k_1'}{\mu}\nabla^2 p_1+\frac{\alpha}{\mu}(p_2-p_1), \qquad \beta_2\frac{\partial p_2}{\partial t}=\frac{k_2'}{\mu}\nabla^2 p_2-\frac{\alpha}{\mu}(p_2-p_1). \tag{2.6}$$

In both of these applications the *cross effects* have been neglected. For example, in Aifantis [3] the more general system arising for flow in fissured rocks is shown to be given by

$$\begin{aligned} A_{11}\frac{\partial p_1}{\partial t}-A_{12}\frac{\partial p_2}{\partial t}&=\bar{D}_1\nabla^2 p_1+\bar{E}_1\nabla^2 p_2+K(p_2-p_1),\\ A_{21}\frac{\partial p_2}{\partial t}-A_{22}\frac{\partial p_1}{\partial t}&=\bar{D}_2\nabla^2 p_2+\bar{E}_2\nabla^2 p_1-K(p_2-p_1), \end{aligned} \tag{2.7}$$

where A_{11}, A_{12}, A_{21}, A_{22}, $\bar{D}_1$, $\bar{D}_2$, $\bar{E}_1$, and $\bar{E}_2$ are constants related to certain physical properties of the problem. This system of equations can be reduced to (1.1) by means of the transformation

$$u_1 = A_{11} p_1 - A_{12} p_2, \qquad u_2 = A_{21} p_2 - A_{22} p_1. \tag{2.8}$$

As a final example of a model also giving rise to (1.1) we consider a model of an arms race originally proposed by Richardson [12] and subsequently extended by Gopalsamy [7]. Richardson [12] proposed that the military spending of two nations locked in an arms race obeys the system

$$\begin{aligned} dp(t)/dt &= -A_1 p(t) + B_1 q(t) + c_1, \\ dq(t)/dt &= -A_2 q(t) + B_2 p(t) + c_2, \end{aligned} \tag{2.9}$$

where $p(t)$ and $q(t)$ denote armament levels of the two nations at time t and A_1, A_2, B_1, B_2, c_1, and c_2 denote positive constants. The constants B_1 and B_2 are called *threat coefficients* signifying to what extent a nation's armament level is increased with respect to the other nation's armament level. Constants A_1 and A_2 measure inhibiting circumstances to armament buildup. Constants c_1 and c_2 measure the circumstances which prevent a complete disarmament if both nations have zero armaments. When the armament levels remain constant for a long time there is a *balance of power*. In such a period the armament levels are given by

$$\begin{aligned} p_0 &= (c_1 A_2 + c_2 B_1)/(A_1 A_2 - B_1 B_2), \\ q_0 &= (c_1 B_2 + c_2 A_1)/(A_1 A_2 - B_1 B_2), \end{aligned} \tag{2.10}$$

and this situation is stable provided $A_1 A_2 - B_1 B_2 > 0$. Gopalsamy [7] assumes that the quality of weapons is represented by a continuous variable x and that weapon deterioration is a stochastic process. In this case $p(x, t)$ and $q(x, t)$ satisfy

$$\begin{aligned} \frac{\partial p}{\partial t} + \mu_1 \frac{\partial p}{\partial x} &= \frac{\bar{\sigma}_1^2}{2} \frac{\partial^2 p}{\partial x^2} - A_1 p + B_1 q + c_1, \\ \frac{\partial q}{\partial t} + \mu_2 \frac{\partial q}{\partial x} &= \frac{\bar{\sigma}_2^2}{2} \frac{\partial^2 q}{\partial x^2} - A_2 q + B_2 p + c_2, \end{aligned} \tag{2.11}$$

where μ_1, μ_2, $\bar{\sigma}_1$, and $\bar{\sigma}_2$ are positive constants. Gopalsamy [7] notes that even when the quality of weapon systems is taken into account the necessary and sufficient condition for the existence and asymptotic stability of balance of power is still $A_1 A_2 - B_1 B_2 > 0$. If we further generalise Richardson's model by assuming that each nation responds to the rate of buildup of the other armaments levels, then instead of (2.9) we have

$$dp(t)/dt = -A_1 p(t) + B_1 q(t) + C_1(dq(t)/dt) + c_1,$$
$$dq(t)/dt = -A_2 q(t) + B_2 p(t) + C_2(dp(t)/dt) + c_2, \tag{2.12}$$

while in place of (2.11) we have

$$\frac{\partial p}{\partial t} + \mu_1 \frac{\partial p}{\partial x} = \frac{\bar{\sigma}_1^2}{2} \frac{\partial^2 p}{\partial x^2} - A_1 p + B_1 q + C_1 \frac{\partial q}{\partial t} + c_1,$$
$$\frac{\partial q}{\partial t} + \mu_2 \frac{\partial q}{\partial x} = \frac{\bar{\sigma}_2^2}{2} \frac{\partial^2 q}{\partial x^2} - A_2 q + B_2 p + C_2 \frac{\partial p}{\partial t} + c_2. \tag{2.13}$$

For the problem of stability of the *balance of power* situation (see Gopalsamy [7]) we examine small deviations from (2.10). Thus with

$$p(x, t) = p_0 + u(x, t), \qquad q(x, t) = q_0 + v(x, t), \tag{2.14}$$

and with the assumption

$$\mu_1 \bar{\sigma}_2^2 = \mu_2 \bar{\sigma}_1^2, \tag{2.15}$$

we obtain a system of equations of the form (1.1) with L in place of ∇^2, where

$$L = (\partial^2/\partial x^2) - \kappa(\partial/\partial x), \tag{2.16}$$

where $\kappa = \mu_i/\bar{\sigma}_i^2$ $(i = 1, 2)$, and $\{u_1, u_2\}$ are defined by

$$u_1(x, t) = u(x, t) - C_1 v(x, t), \qquad u_2(x, t) = v(x, t) - C_2 u(x, t). \tag{2.17}$$

Although we shall not consider any of these applications in any detail, the above serves to illustrate the variety of physical models which result in coupled systems of parabolic equations of the type (1.1). We also note that the general solutions given in Theorem 3 of Section 3 can be employed for (1.1) with any spatial operator L in place of ∇^2 provided that we interpret $\{h_1, h_2\}$ as solutions of

$$\partial h/\partial t = L(h), \tag{2.18}$$

so that, for example, we can make use of these formulae for the model of the arms race discussed above.

3. Summary of Main Results

The main results of this paper are: uniqueness theorems for the coupled system (1.1) and the fourth order equation (1.3), the solutions of (1.1) in terms of solutions of the classical diffusion equation (1.10), and the solutions of (1.10) in terms of the solutions of (1.1). We first need a definition.

Definition. Solutions $\{u_1, u_2\} \in H$ in S if and only if:

(i) $\{u_1, u_2\} \in C^2$ in S, and

(ii) $\{u_1, u_2\}$ satisfy (1.1) in S,

where S is an arbitrary region of the $\mathbf{x} - t$ space, and C^2 denotes the set of functions which are continuous together with their derivatives up to second order.

Theorem 1. *Given that Ω is a bounded domain inside a piecewise continuously differentiable surface $\partial\Omega$, $T > 0$, $\mathbb{R} = \{(\mathbf{x}, t): \mathbf{x} \in \Omega \cup \partial\Omega$, $0 \leqslant t \leqslant T\}$*, and if:

(i) $\{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\} \in H$ *if* $(\mathbf{x}, t) \in \mathbb{R}$,

(ii) $\{u_1(\mathbf{x}, 0), u_2(\mathbf{x}, 0)\} = 0$ *if* $\mathbf{x} \in \Omega \cup \partial\Omega$,

(iii) $\{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\} = 0$ *if* $\mathbf{x} \in \partial\Omega$, $0 \leqslant t \leqslant T$,

(iv) $A_1, A_2, B_1, B_2 \geqslant 0$, $A_1 A_2 - B_1 B_2 \geqslant 0$,
$D_1, D_2, E_1, E_2 \geqslant 0$, $D_1 D_2 - E_1 E_2 \geqslant 0$,

(v) $$2E_1 E_2 (A_1 A_2 - B_1 B_2)^{1/2} \geqslant |B_1 E_2 - B_2 E_1| \, (D_1 D_2)^{1/2} - (B_1 E_2 + B_2 E_1)(D_1 D_2 - E_1 E_2)^{1/2},$$

then $\{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\} = 0$ *when* $(\mathbf{x}, t) \in \mathbb{R}$.

For the fourth order equation (1.3) we need

Definition. The solution $u \in H_1$ in S if and only if:

(i) $u \in C^4$ in S, and

(ii) u satisfies (1.3) in S,

where S is an arbitrary region of the $\mathbf{x} - t$ space, and C^4 denotes the set of functions continuous together with their derivatives up to fourth order.

Theorem 2. *Given that Ω is a bounded domain inside a piecewise continuously differentiable surface $\partial\Omega$, $T > 0$, $\mathbb{R} = \{(\mathbf{x}, t): \mathbf{x} \in \Omega \cup \partial\Omega$, $0 \leqslant t \leqslant T\}$, and if:*

(i) $u(\mathbf{x}, t) \in H_1$ *if* $(\mathbf{x}, t) \in \mathbb{R}$,

(ii) *either* $(\partial u/\partial t)(\mathbf{x}, t) = (\partial^2 u/\partial t \partial n)(\mathbf{x}, t) = 0$ *if* $\mathbf{x} \in \partial\Omega$ *or* $(\partial u/\partial t)(\mathbf{x}, t) = \nabla^2(\partial u/\partial t)(\mathbf{x}, t) = 0$ *if* $\mathbf{x} \in \partial\Omega$,

(iii) $u(\mathbf{x}, 0) = (\partial u/\partial t)(\mathbf{x}, 0) = 0$ *if* $\mathbf{x} \in \Omega \cup \partial\Omega$,

(iv) $\nabla u(\mathbf{x}, 0) = 0$ *if* $\mathbf{x} \in \Omega \cup \partial\Omega$,

(v) $A_1 + A_2 \geqslant 0$, $D_1 + D_2 \geqslant 0$, $D_1 D_2 - E_1 E_2 \geqslant 0$, $A_1 A_2 - B_1 B_2 \geqslant 0$, $A_1 D_2 + A_2 D_1 + B_1 E_2 + B_2 E_1 \geqslant 0$,

then $u(\mathbf{x}, t) = 0$ *whenever* $(\mathbf{x}, t) \in \mathbb{R}$, *and* $\mathbf{n}$ *denotes the unit normal of* $\partial\Omega$.

The solutions of (1.1) are written in terms of solutions of the classical diffusion equation in Theorem 3.

THEOREM 3. *If* $\{h_1, h_2\}$ *are solutions of* (1.10) *subject to the initial conditions*

$$h_1(\mathbf{x}, 0) = f_1(\mathbf{x}), \qquad h_2(\mathbf{x}, 0) = f_2(\mathbf{x}), \tag{3.1}$$

then the solutions of (1.1) *subject to the initial conditions*

$$u_1(\mathbf{x}, 0) = f_1(\mathbf{x}), \qquad u_2(\mathbf{x}, 0) = f_2(\mathbf{x}), \tag{3.2}$$

are given by

$$\begin{aligned} u_1(\mathbf{x}, t) = {} & (2\alpha)^{-1} \{e^{-\nu_2 t}[(D_2 - d_2)\, h_1(\mathbf{x}, d_2 t) - E_1 h_2(\mathbf{x}, d_2 t)] \\ & - e^{-\nu_1 t}[(D_2 - d_1)\, h_1(\mathbf{x}, d_1 t) - E_1 h_2(\mathbf{x}, d_1 t)]\} \\ & + e^{\delta t} \int_{d_2 t}^{d_1 t} [K_1(t, \xi)\, h_1(\mathbf{x}, \xi) + K_2(t, \xi)\, h_2(\mathbf{x}, \xi)]\, e^{-\gamma \xi}\, d\xi, \\ u_2(\mathbf{x}, t) = {} & (2\alpha)^{-1} \{e^{-\nu_2 t}\, [(D_1 - d_2)\, h_2(\mathbf{x}, d_2 t) - E_2 h_1(\mathbf{x}, d_2 t)] \\ & - e^{-\nu_1 t}[(D_1 - d_1)\, h_2(\mathbf{x}, d_1 t) - E_2 h_1(\mathbf{x}, d_1 t)]\} \\ & + e^{\delta t} \int_{d_2 t}^{d_1 t} [K_3(t, \xi)\, h_2(\mathbf{x}, \xi) + K_4(t, \xi)\, h_1(\mathbf{x}, \xi)]\, e^{-\gamma \xi}\, d\xi, \end{aligned} \tag{3.3}$$

where $K_i(t, \xi)$ $(i = 1, 2, 3, 4)$, *are defined by*

$$\begin{aligned} K_1(t, \xi) = {} & [\beta(D_1 - D_2) - \alpha(A_1 - A_2)] \frac{I_0(\eta)}{4\alpha^2} \\ & + \frac{\{4\alpha^2 t - (D_1 - D_2)[(D_1 + D_2)\, t - 2\xi]\}\, k^{1/2} I_1(\eta)}{8\alpha^2 [(d_1 t - \xi)(\xi - d_2 t)]^{1/2}}, \end{aligned}$$

$$K_2(t, \xi) = (\alpha B_1 + \beta E_1)\frac{I_0(\eta)}{2\alpha^2} - \frac{E_1[(D_1 + D_2)t - 2\xi]\, k^{1/2} I_1(\eta)}{4\alpha^2[(d_1 t - \xi)(\xi - d_2 t)]^{1/2}},$$

$$K_3(t, \xi) = -[\beta(D_1 - D_2) - \alpha(A_1 - A_2)]\frac{I_0(\eta)}{4\alpha^2} + \frac{\{4\alpha^2 t + (D_1 - D_2)[(D_1 + D_2)t - 2\xi]\}\, k^{1/2} I_1(\eta)}{8\alpha^2[(d_1 t - \xi)(\xi - d_2 t)]^{1/2}},$$

$$K_4(t, \xi) = (\alpha B_2 + \beta E_2)\frac{I_0(\eta)}{2\alpha^2} - \frac{E_2[(D_1 + D_2)t - 2\xi]\, k^{1/2} I_1(\eta)}{4\alpha^2[(d_1 t - \xi)(\xi - d_2 t)]^{1/2}}, \tag{3.4}$$

and where α, β, γ, δ, d_1, d_2, ν_1, ν_2, k, *and* η *are given by*

$$\alpha = 2^{-1}[(D_1 - D_2)^2 + 4E_1E_2]^{1/2}, \tag{3.5a}$$

$$\beta = 2^{-1}[(A_1 - A_2)(D_1 - D_2) - 2(B_1E_2 + B_2E_1)][(D_1 - D_2)^2 + 4E_1E_2]^{-1/2}, \tag{3.5b}$$

$$\gamma = [(A_1 - A_2)(D_1 - D_2) - 2(B_1E_2 + B_2E_1)][(D_1 - D_2)^2 + 4E_1E_2]^{-1}, \tag{3.5c}$$

$$\delta = 2^{-1}\{-(A_1 + A_2) + (D_1 + D_2)[(A_1 - A_2)(D_1 - D_2) - 2(B_1E_2 + B_2E_1)][(D_1 - D_2)^2 + 4E_1E_2]^{-1}\}, \tag{3.5d}$$

$$d_1 = 2^{-1}\{(D_1 + D_2) + [(D_1 - D_2)^2 + 4E_1E_2]^{1/2}\}, \tag{3.5e}$$

$$d_2 = 2^{-1}\{(D_1 + D_2) - [(D_1 - D_2)^2 + 4E_1E_2]^{1/2}\}, \tag{3.5f}$$

$$\nu_1 = 2^{-1}\{(A_1 + A_2) + [(A_1 - A_2)(D_1 - D_2) - 2(B_1E_2 + B_2E_1)][(D_1 - D_2)^2 + 4E_1E_2]^{-1/2}\}, \tag{3.5g}$$

$$\nu_2 = 2^{-1}\{(A_1 + A_2) - [(A_1 - A_2)(D_1 - D_2) - 2(B_1E_2 + B_2E_1)][(D_1 - D_2)^2 + 4E_1E_2]^{-1/2}\}, \tag{3.5h}$$

$$k = [E_1E_2(A_1 - A_2)^2 + B_1B_2(D_1 - D_2)^2 - (B_1E_2 - B_2E_1)^2 + (A_1 - A_2)(D_1 - D_2)(B_1E_2 + B_2E_1)][(D_1 - D_2)^2 + 4E_1E_2]^{-1}, \tag{3.5i}$$

$$\eta = k^{1/2}\alpha^{-1}[(d_1 t - \xi)(\xi - d_2 t)]^{1/2}. \tag{3.5j}$$

Theorem 4 is the inverse counterpart of Theorem 3. The roles of $\{u_1, u_2\}$ and $\{h_1, h_2\}$ are reversed and the similarity between the expressions for $\{u_1, u_2\}$ in Theorem 3 and $\{h_1, h_2\}$ in Theorem 4 should be noted. Theorem 4 formally enables prescribed boundary data on $\{u_1, u_2\}$ to be transferred to $\{h_1, h_2\}$.

THEOREM 4. *With $\{h_1, h_2\}$ and $\{u_1, u_2\}$ defined as in Theorem 3, we have*

$$\begin{aligned} h_1(\mathbf{x}, t) = {} & (2\alpha)^{-1} \{e^{v_1 t/d_1}[(D_1 - d_2) u_1(\mathbf{x}, d_1^{-1}t) + E_1 u_2(\mathbf{x}, d_1^{-1}t)] \\ & - e^{v_2 t/d_2}[(D_1 - d_1) u_1(\mathbf{x}, d_2^{-1}t) + E_1 u_2(\mathbf{x}, d_2^{-1}t)]\} \\ & + e^{\gamma t} \int_{d_1^{-1}t}^{d_2^{-1}t} [L_1(t, \xi) u_1(\mathbf{x}, \xi) + L_2(t, \xi) u_2(\mathbf{x}, \xi)] e^{-\delta\xi} d\xi, \\ h_2(\mathbf{x}, t) = {} & (2\alpha)^{-1} \{e^{v_1 t/d_1}[(D_2 - d_2) u_2(\mathbf{x}, d_1^{-1}t) + E_2 u_1(\mathbf{x}, d_1^{-1}t)] \\ & - e^{v_2 t/d_2}[(D_2 - d_1) u_2(\mathbf{x}, d_2^{-1}t) + E_2 u_1(\mathbf{x}, d_2^{-1}t)]\} \\ & + e^{\gamma t} \int_{d_1^{-1}t}^{d_2^{-1}t} [L_3(t, \xi) u_2(\mathbf{x}, \xi) + L_4(t, \xi) u_1(\mathbf{x}, \xi)] e^{-\delta\xi} d\xi, \end{aligned} \tag{3.6}$$

where $L_i(t, \xi)$ $(i = 1, 2, 3, 4)$, are defined by

$$\begin{aligned} L_1(t, \xi) = {} & \frac{\theta_1 I_0(\eta^*)}{2\alpha} + \frac{(kc)^{1/2} I_1(\eta^*)}{4\alpha^2} \\ & \times \left[(d_1 - D_1) \left(\frac{\xi - d_1^{-1}t}{d_2^{-1}t - \xi}\right)^{1/2} - (d_2 - D_1) \left(\frac{d_2^{-1}t - \xi}{\xi - d_1^{-1}t}\right)^{1/2}\right], \\ L_2(t, \xi) = {} & \frac{\theta_4 I_0(\eta^*)}{2\alpha} + \frac{(kc)^{1/2} E_1 I_1(\eta^*)}{4\alpha^2} \left[\left(\frac{d_2^{-1}t - \xi}{\xi - d_1^{-1}t}\right)^{1/2} - \left(\frac{\xi - d_1^{-1}t}{d_2^{-1}t - \xi}\right)^{1/2}\right], \\ L_3(t, \xi) = {} & \frac{\theta_7 I_0(\eta^*)}{2\alpha} + \frac{(kc)^{1/2} I_1(\eta^*)}{4\alpha^2} \\ & \times \left[(d_1 - D_2) \left(\frac{\xi - d_1^{-1}t}{d_2^{-1}t - \xi}\right)^{1/2} - (d_2 - D_2) \left(\frac{d_2^{-1}t - \xi}{\xi - d_1^{-1}t}\right)^{1/2}\right], \\ L_4(t, \xi) = {} & \frac{\theta_{10} I_0(\eta^*)}{2\alpha} + \frac{(kc)^{1/2} E_2 I_1(\eta^*)}{4\alpha^2} \left[\left(\frac{d_2^{-1}t - \xi}{\xi - d_1^{-1}t}\right)^{1/2} - \left(\frac{\xi - d_1^{-1}t}{d_2^{-1}t - \xi}\right)^{1/2}\right], \end{aligned} \tag{3.7}$$

and θ_1, θ_4, θ_7, and θ_{10} are constants which are given by (8.23). *Further, the expression η^* is defined by*

$$\eta^* = (kc)^{1/2} \alpha^{-1}[(\xi - d_1^{-1}t)(d_2^{-1}t - \xi)]^{1/2}, \tag{3.8}$$

where c is given by $(5.2)_3$ and the other constants appearing in (3.6)–(3.8) *are given by* (3.5).

4. Uniqueness of Solutions

We now consider the uniqueness question for the coupled system (1.1) and the fourth order system (1.3). The proof of Theorem 1 is as follows: We multiply $(1.1)_1$ by u_1 and $(1.1)_2$ by u_2 and use the identity

$$f\nabla^2 g = \nabla \cdot (f\nabla g) + \tfrac{1}{2}|\nabla(f-g)|^2 - \tfrac{1}{2}|\nabla f|^2 - \tfrac{1}{2}|\nabla g|^2. \tag{4.1}$$

The resultant equations are integrated over Ω and the divergence theorem is applied to the volume integrals to obtain surface integrals. The equations we deduce are

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u_1^2\,dS = \int_{\partial\Omega} &\left(D_1\frac{\partial u_1}{\partial n} + E_1\frac{\partial u_2}{\partial n}\right) u_1\,ds \\
&- \int_\Omega (D_1 + E_1/2)\,|\nabla u_1|^2\,dS - \frac{1}{2}\int_\Omega E_1\,|\nabla u_2|^2\,dS \\
&+ \frac{1}{2}\int_\Omega E_1\,|\nabla(u_1 - u_2)|^2\,dS + \int_\Omega (B_1 u_1 u_2 - A_1 u_1^2)\,dS, \\
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u_2^2\,dS = \int_{\partial\Omega} &\left(D_2\frac{\partial u_2}{\partial n} + E_2\frac{\partial u_1}{\partial n}\right) u_2\,ds \\
&- \int_\Omega (D_2 + E_2/2)\,|\nabla u_2|^2\,dS - \frac{1}{2}\int_\Omega E_2\,|\nabla u_1|^2\,dS \\
&+ \frac{1}{2}\int_\Omega E_2\,|\nabla(u_1 - u_2)|^2\,dS + \int_\Omega (B_2 u_1 u_2 - A_2 u_2^2)\,dS,
\end{aligned} \tag{4.2}$$

where $\mathbf{n}$ denotes the unit normal of $\partial\Omega$, dS denotes elementary volume, and ds denotes elementary area. Multiplying $(4.2)_1$ by A and $(4.2)_2$ by B and adding, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}&\int_\Omega (Au_1^2 + Bu_2^2)\,dS \\
&= \int_\Omega [A(B_1u_1u_2 - A_1u_1^2) + B(B_2u_1u_2 - A_2u_2^2)]\,dS \\
&\quad - \int_\Omega (AD_1\,|\nabla u_1|^2 + BD_2\,|\nabla u_2|^2)\,dS \\
&\quad + \frac{1}{2}(AE_1 + BE_2)\int_\Omega [|\nabla u_1|^2 + |\nabla u_2|^2 - |\nabla(u_1 - u_2)|^2]\,dS,
\end{aligned} \tag{4.3}$$

where the surface integrals are zero by boundary condition (iii). In order to show that the integrand on the left-hand side of (4.3) is nonpositive we introduce $\mathbf{v}_1 = \nabla u_1$ and $\mathbf{v}_2 = \nabla u_2$ with $v_1 = |\mathbf{v}_1|$ and $v_2 = |\mathbf{v}_2|$. We consider the integrands of the second and third integrals on the right-hand side of (4.3). If A and B are positive constants, then using the triangle inequality these integrands can be shown (see Lee [10]) to be less than or equal to the quantity

$$Q_1 = -AD_1 \left[v_1 - \frac{v_2(AE_1 + BE_2)}{2AD_1} \right]^2 - \left[BD_2 - \frac{(AE_1 + BE_2)^2}{4AD_1} \right] v_2^2. \tag{4.4}$$

The integrand of the first integral on the right-hand side of (4.3) is equal to the quantity

$$Q_2 = -AA_1 \left[u_1 - \frac{u_2(AB_1 + BB_2)}{2AA_1} \right]^2 - \left[BA_2 - \frac{(AB_1 + BB_2)^2}{4AA_1} \right] u_2^2. \tag{4.5}$$

The quantities Q_1 and Q_2 are negative if A and B are positive constants which satisfy the following inequalities:

$$4ABD_1D_2 \geqslant (AE_1 + BE_2)^2, \qquad 4ABA_1A_2 \geqslant (AB_1 + BB_2)^2. \tag{4.6}$$

Considering (4.6) as equalities and defining $x = A/B$ we can obtain (see Lee [10])

$$x = (B_2 + E_2\bar{w})(B_1 + E_1\bar{w})^{-1}, \tag{4.7}$$

where $\bar{w}$ is defined by

$$\bar{w} = [(A_1A_2 - B_1B_2)(D_1D_2 - E_1E_2)^{-1}]^{1/2} \geqslant 0. \tag{4.8}$$

Substituting from (4.7) and (4.8) into either of (4.6) one can obtain the inequality

$$4(B_1 + E_1\bar{w})(B_2 + E_2\bar{w})(D_1D_2 - E_1E_2) \geqslant (B_1E_2 - B_2E_1)^2. \tag{4.9}$$

This inequality can be written as

$$(\bar{w} + \bar{\alpha} + \bar{\beta})(\bar{w} + \bar{\alpha} - \bar{\beta}) \geqslant 0, \tag{4.10}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are constants defined by

$$\begin{aligned} \bar{\alpha} &= (B_1E_2 + B_2E_1)/2E_1E_2, \\ \bar{\beta} &= (B_1E_2 - B_2E_1)(D_1D_2)^{1/2}/2E_1E_2(D_1D_2 - E_1E_2)^{1/2}. \end{aligned} \tag{4.11}$$

Inequality (4.10) implies that both terms contained in the brackets must be greater than or equal to zero. If both were negative, then $\bar{w} < -\bar{\alpha} < 0$, contradicting (4.8). Hence

$$\bar{w} \geqslant -(B_1E_2 + B_2E_1)/2E_1E_2 + |B_1E_2 - B_2E_1|\,(D_1D_2)^{1/2}/2E_1E_2(D_1D_2 - E_1E_2)^{1/2}. \tag{4.12}$$

Upon multiplying (4.12) by $2E_1E_2(D_1D_2 - E_1E_2)^{1/2}$ and using (4.8) for $\bar{w}$ we obtain

$$2E_1E_2(A_1A_2 - B_1B_2)^{1/2} \geqslant |B_1E_2 - B_2E_1|\,(D_1D_2)^{1/2} - (B_1E_2 + B_2E_1)(D_1D_2 - E_1E_2)^{1/2}. \tag{4.13}$$

We notice that if $D_1D_2 = E_1E_2$, then we can still deduce (4.13) directly from (4.6) in a straightforward manner. Thus, inequality (4.13) is certainly a sufficient condition for finding positive constants A and B satisfying (4.6), and is therefore a sufficient condition for the right-hand side of (4.3) to be negative. The integral on the left-hand side of (4.3) vanishes at $t = 0$ by the initial condition (ii). Hence we obtain the inequality

$$\int_\Omega (Au_1^2 + Bu_2^2)\,dS \leqslant 0, \qquad t > 0. \tag{4.14}$$

Since the integrand of this integral is nonnegative we have by continuity that $\{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t)\} = 0$ whenever $(\mathbf{x}, t) \in \mathbb{R}$, which concludes the proof of Theorem 1.

We remark that this theorem is also true if $E_1 = E_1 = 0$ (see Aifantis and Hill [4]). Uniqueness is also guaranteed if boundary conditions (iii) are replaced by

$$\left\{\frac{\partial u_1}{\partial n}(\mathbf{x}, t), \frac{\partial u_2}{\partial n}(\mathbf{x}, t)\right\} = 0, \qquad \mathbf{x} \in \partial\Omega. \tag{4.15}$$

Theorem 2 is proved in much the same way is Theorem 1. The proof uses (4.1), where f and g are replaced by u or $\partial u/\partial t$. In this case there is no additional condition on the constants corresponding to (4.13). After integrating (1.3) over Ω and using (4.1) along with the boundary conditions (ii), denoting $\partial u/\partial t$ by u_t, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega [u_t^2 + d\,|\nabla u|^2 + eu^2]\,dS = -a\int_\Omega u_t^2\,dS - b\int_\Omega |\nabla u_t|^2\,dS - c\int_\Omega (\nabla^2 u_t)^2 dS, \tag{4.16}$$

where a–e are constants defined by (5.2). Using the usual technique in such uniqueness problems (see Weinberger [15]) we can go on to show that $u(\mathbf{x}, t) = 0$. This concludes the proofs of the uniqueness theorems.

5. Source Solutions

To demonstrate Theorems 3 and 4 we first consider the intermediate step of obtaining the derivations of the source solutions of (1.1). These are solutions which vanish at infinity and satisfy initial conditions (1.7). In the following, $\hat{u}(\mathbf{x}, s)$ denotes the Laplace transform of $u(\mathbf{x}, t)$ with respect to t, $\bar{u}(\mathbf{w}, t)$ denotes the Fourier transform of $u(\mathbf{x}, t)$ with respect to $\mathbf{x}$, and $U(\mathbf{w}, s)$ denotes that both transforms have been performed on $u(\mathbf{x}, t)$. Upon taking both transforms of (1.1) and applying (1.7) we obtain

$$U_1(\mathbf{w}, s) = \frac{\rho_1 s + (D_2\rho_1 - E_1\rho_2) w^2 + (B_1\rho_2 + A_2\rho_1)}{s^2 + (aw^2 + b) s + cw^4 + dw^2 + e},$$

$$U_2(\mathbf{w}, s) = \frac{\rho_2 s + (D_1\rho_2 - E_2\rho_1) w^2 + (B_2\rho_1 + A_1\rho_2)}{s^2 + (aw^2 + b) s + cw^4 + dw^2 + e}, \tag{5.1}$$

where $w^2 = \sum_{i=1}^{n} w_i^2$, $n = 1, 2$, or 3 being the dimension of $\mathbf{w}$, and where a, b, c, d, and e are constants defined by

$$a = D_1 + D_2, \qquad b = A_1 + A_2, \qquad c = D_1 D_2 - E_1 E_2,$$

$$d = A_1 D_2 + A_2 D_1 + B_1 E_2 + B_2 E_1, \qquad e = A_1 A_2 - B_1 B_2. \tag{5.2}$$

Let $s_1(w)$ and $s_2(w)$ denote the roots of the denominator in (5.1). We then have

$$s_1(w) = -\lambda + (\mu^2 + k)^{1/2}, \qquad s_2(w) = -\lambda - (\mu^2 + k)^{1/2}, \tag{5.3}$$

where λ, μ, and k are defined by

$$\lambda = \frac{1}{2}(aw^2 + b), \qquad \mu = \frac{(a^2 - 4c)^{1/2}}{2}\left[w^2 - \left(\frac{2d - ab}{a^2 - 4c}\right)\right],$$

$$k = \frac{1}{4}\left[(b^2 - 4e) - \frac{(2d - ab)^2}{(a^2 - 4c)}\right]. \tag{5.4}$$

At this stage it should be noted that constraints on the constants appearing in (1.1) can be determined by requiring the roots of

$$s^2 + (aw^2 + b) s + cw^4 + dw^2 + e = 0, \tag{5.5}$$

are real. This corresponds physically to assuming exponential decay of the source solutions. Sufficient conditions for the descriminant of (5.5) to be positive are (1.8) and

$$E_1 E_2 > -\tfrac{1}{4}(D_1 - D_2)^2, \qquad B_1 B_2 > -\tfrac{1}{4}(A_1 - A_2)^2. \tag{5.6}$$

Since B_i, $E_i \geqslant 0$ $(i = 1, 2)$, inequalities (5.6) are clearly satisfied and (1.8) implies $k > 0$. Inverting (5.1) with respect to s we obtain

$$\begin{aligned} \bar{u}_1(\mathbf{w}, t) &= e^{-\lambda t}\{\rho_1 \cosh[t(\mu^2 + k)^{1/2}] - [\rho_1(\lambda - D_2 w^2 - A_2) \\ &\quad - \rho_2(B_1 - E_1 w^2)](\mu^2 + k)^{-1/2} \sinh[t(\mu^2 + k)^{1/2}]\}, \\ \bar{u}_2(\mathbf{w}, t) &= e^{-\lambda t}\{\rho_2 \cosh[t(\mu^2 + k)^{1/2}] - [\rho_2(\lambda - D_1 w^2 - A_1) \\ &\quad - \rho_1(B_2 - E_2 w^2)](\mu^2 + k)^{-1/2} \sinh[t(\mu^2 + k)^{1/2}]\}, \end{aligned} \tag{5.7}$$

where we have used

$$s_1 - s_2 = 2(\mu^2 + k)^{1/2}. \tag{5.8}$$

We define $\bar{p}_1(\mathbf{w}, t)$ and $\bar{p}_2(\mathbf{w}, t)$ by

$$\begin{aligned} \bar{p}_1(\mathbf{w}, t) &= e^{-\lambda t}(\mu^2 + k)^{-1/2} \sinh[t(\mu^2 + k)^{1/2}], \\ \bar{p}_2(\mathbf{w}, t) &= w^2 \bar{p}_1(\mathbf{w}, t). \end{aligned} \tag{5.9}$$

Then, from (5.7) and (5.9) we obtain

$$\begin{aligned} \bar{u}_1(\mathbf{w}, t) &= \rho_1(\partial \bar{p}_1/\partial t) + (\rho_2 B_1 + \rho_1 A_2)\bar{p}_1 + (\rho_1 D_2 - \rho_2 E_1)\bar{p}_2, \\ \bar{u}_2(\mathbf{w}, t) &= \rho_2(\partial \bar{p}_1/\partial t) + (\rho_1 B_2 + \rho_2 A_1)\bar{p}_1 + (\rho_2 D_1 - \rho_1 E_2)\bar{p}_2. \end{aligned} \tag{5.10}$$

From (5.9) we deduce, using Sonine's second finite integral [14], and with $k > 0$

$$\begin{aligned} \bar{p}_1(\mathbf{w}, t) &= t e^{-\lambda t} \int_0^{\pi/2} I_0(k^{1/2} t \sin\theta) \cosh(\mu t \cos\theta) \sin\theta \, d\theta, \\ \bar{p}_2(\mathbf{w}, t) &= t e^{-\lambda t} w^2 \int_0^{\pi/2} I_0(k^{1/2} t \sin\theta) \cosh(\mu t \cos\theta) \sin\theta \, \mathrm{d}\theta. \end{aligned} \tag{5.11}$$

In one dimension the inversion formula

$$p_j(x, t) = \frac{1}{\pi} \int_0^\infty \bar{p}_j(w, t) \cos(wx)\, dw \qquad (j = 1, 2), \tag{5.12}$$

can be used with (5.11) to obtain

$$p_1(x,t)=\frac{e^{-bt/2}}{4\pi^{1/2}}\int_{-t}^{t}\frac{e^{\beta v}I_0[k^{1/2}(t^2-v^2)^{1/2}]\,e^{-x^2/(2at-4\alpha v)}}{(at/2-\alpha v)^{1/2}}\,dv,$$

$$p_2(x,t)=\frac{e^{-bt/2}}{8\pi^{1/2}}\int_{-t}^{t}\frac{e^{\beta v}I_0[k^{1/2}(t^2-v^2)^{1/2}]}{(at/2-\alpha v)^{3/2}}\left[1-\frac{x^2}{(at-\alpha v)}\right]e^{-x^2/(2at-4\alpha v)}\,dv, \tag{5.13}$$

where α and β are constants defined by (3.5a) and (3.5b), respectively. To obtain (5.13) from (5.11) and (5.12) it is required that

$$a+\alpha\cos\theta>0,\qquad a-\alpha\cos\theta>0, \tag{5.14}$$

so that certain integrals converge. From (5.14) we deduce the following sufficient conditions on the fundamental constants such that (5.13) is true,

$$D_1+D_2>0,\qquad D_1D_2-E_1E_2>0,\qquad E_1E_2>-\tfrac{1}{4}(D_1-D_2)^2. \tag{5.15}$$

Notice that the right most inequality of (5.15) is the same as the left-hand inequality of (5.6). From (5.10) and (5.13) the source solutions can be deduced,

$$\begin{aligned}u_1(x,t)={}&\frac{[\rho_1(D_2-d_2)-\rho_2E_1]\,e^{-v_2t}e^{-x^2/(4d_2t)}}{4\alpha(\pi d_2t)^{1/2}}\\&-\frac{[\rho_1(D_2-d_1)-\rho_2E_1]\,e^{-v_1t}e^{-x^2/(4d_1t)}}{4\alpha(\pi d_1t)^{1/2}}\\&+e^{\delta t}\int_{d_2t}^{d_1t}[\rho_1K_1(t,\xi)+\rho_2K_2(t,\xi)]\,e^{-\gamma\xi}\frac{e^{-x^2/(4\xi)}}{2(\pi\xi)^{1/2}}\,d\xi,\\u_2(x,t)={}&\frac{[\rho_2(D_1-d_2)-\rho_1E_2]\,e^{-v_2t}e^{-x^2/(4d_2t)}}{4\alpha(\pi d_2t)^{1/2}}\\&-\frac{[\rho_2(D_1-d_1)-\rho_1E_2]\,e^{-v_1t}e^{-x^2/(4d_1t)}}{4\alpha(\pi d_1t)^{1/2}}\\&+e^{\delta t}\int_{d_2t}^{d_1t}[\rho_2K_3(t,\xi)+\rho_1K_4(t,\xi)]\,e^{-\gamma\xi}\frac{e^{-x^2/(4\xi)}}{2(\pi\xi)^{1/2}}\,d\xi,\end{aligned} \tag{5.16}$$

where $K_i(t,\xi)$ $(i=1,2,3,4)$, are defined by (3.4) while the various constants and η are given by (3.5). Similar formulae for two and three dimensions can be deduced. In place of (5.12) we use

$$p_j(r,t)=\frac{1}{2\pi}\int_0^\infty \bar{p}_j(\mathbf{w},t)\,J_0(wr)\,w\,dw\qquad(j=1,2), \tag{5.17}$$

for two dimensions, while for three dimensions we have

$$p_j(R, t) = \frac{1}{2\pi^2 R} \int_0^\infty \bar{p}_j(\mathbf{w}, t) \sin(wR)\, w\, dw \qquad (j = 1, 2). \tag{5.18}$$

6. Asymptotic Expansions

In this section asymptotic expansions are obtained for the source solutions (5.16) for both $t \to \infty$ and $t \to 0$. In (5.3) we make the substitution

$$w = \rho t^{-1/2}, \tag{6.1}$$

to obtain, for large t,

$$s_1 = -\varepsilon_1 - \rho^2 t^{-1} \sigma_1 + O(t^{-2}), \qquad s_2 = -\varepsilon_2 - \rho^2 t^{-1} \sigma_2 + O(t^{-2}), \tag{6.2}$$

where ε_i, σ_i $(i = 1, 2)$, are constants defined by

$$\begin{aligned}
\varepsilon_1 &= 2^{-1}\{(A_1 + A_2) - [(A_1 - A_2)^2 + 4B_1 B_2]^{1/2}\},\\
\varepsilon_2 &= 2^{-1}\{(A_1 + A_2) + [(A_1 - A_2)^2 + 4B_1 B_2]^{1/2}\},\\
\sigma_1 &= 2^{-1}\{(D_1 + D_2) + [2(B_1 E_2 + B_2 E_1) - (A_1 - A_2)(D_1 - D_2)]\\
&\qquad \times [(A_1 - A_2)^2 + 4B_1 B_2]^{-1/2}\},\\
\sigma_2 &= 2^{-1}\{(D_1 + D_2) - [2(B_1 E_2 + B_2 E_1) - (A_1 - A_2)(D_1 - D_2)]\\
&\qquad \times [(A_1 - A_2)^2 + 4B_1 B_2]^{-1/2}\}.
\end{aligned} \tag{6.3}$$

Using (6.2) for s_1 and s_2 we can obtain the asymptotic expansions for $\{u_1, u_2\}$. The reader is referred to Copson [6] for the usual methods. We use (5.7) and the formula,

$$u_j(x, t) = \frac{1}{\pi} \int_0^\infty \bar{u}_j(w, t) \cos(wx)\, dw \qquad (j = 1, 2), \tag{6.4}$$

to obtain

$$\begin{aligned}
u_1(x, t) &= [\rho_1(A_2 - \varepsilon_1) + \rho_2 B_1](4\pi t \sigma_1)^{-1/2} (\varepsilon_2 - \varepsilon_1)^{-1} e^{-\varepsilon_1 t} e^{-x^2/(4\sigma_1 t)}\\
&\quad - [\rho_1(A_2 - \varepsilon_2) + \rho_2 B_1](4\pi t \sigma_2)^{-1/2} (\varepsilon_2 - \varepsilon_1)^{-1} e^{-\varepsilon_2 t} e^{-x^2/(4\sigma_2 t)}\\
&\quad + O(t^{-3/2}),\\
u_2(x, t) &= [\rho_2(A_1 - \varepsilon_1) + \rho_1 B_2](4\pi t \sigma_1)^{-1/2} (\varepsilon_2 - \varepsilon_1)^{-1} e^{-\varepsilon_1 t} e^{-x^2/(4\sigma_1 t)}\\
&\quad - [\rho_2(A_1 - \varepsilon_2) + \rho_1 B_2](4\pi t \sigma_2)^{-1/2} (\varepsilon_2 - \varepsilon_1)^{-1} e^{-\varepsilon_2 t} e^{-x^2/(4\sigma_2 t)}\\
&\quad + O(t^{-3/2}).
\end{aligned} \tag{6.5}$$

The first terms of (6.5) will dominate as $t \to \infty$ (since $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$). Adding these dominant terms we find that the total concentration, $u = u_1 + u_2$, for large t is asymptotically given by

$$u(x,t) \sim [\rho_1(A_2 + B_2 - \varepsilon_1) + \rho_2(A_1 + B_1 - \varepsilon_1)] \\ \times (4\pi t\sigma_1)^{-1/2} (\varepsilon_2 - \varepsilon_1)^{-1} e^{-\varepsilon_1 t} e^{-x^2/(4\sigma_1 t)}, \tag{6.6}$$

and this will be of the classical diffusion type if

$$A_1 A_2 = B_1 B_2. \tag{6.7}$$

If this is the case the total concentration is then asymptotically of classical diffusion type with diffusivity D^* given by (1.4), and strength $(\rho_1 + \rho_2)$. If we make the substitution (6.1) into (5.3) for small times $(t \to 0)$ we obtain

$$s_1 = -v_2 - \rho^2 t^{-1} d_2 + O(t), \qquad s_2 = -v_1 - \rho^2 t^{-1} d_1 + O(t), \tag{6.8}$$

where v_i, d_i $(i = 1, 2)$, are constants defined by (3.5). Using (5.7), (6.4), and (6.8) the asymptotic expansins as $t \to 0$ for $\{u_1, u_2\}$ can be obtained,

$$\begin{aligned} u_1(x,t) &= [\rho_1(D_2 - d_2) - \rho_2 E_1](4\pi t d_2)^{-1/2} (2\alpha)^{-1} e^{-v_2 t} e^{-x^2/(4d_2 t)} \\ &\quad - [\rho_1(D_2 - d_1) - \rho_2 E_1](4\pi t d_1)^{-1/2} (2\alpha)^{-1} e^{-v_1 t} e^{-x^2/(4d_1 t)} + O(t^{1/2}), \\ u_2(x,t) &= [\rho_2(D_1 - d_2) - \rho_1 E_2](4\pi t d_2)^{-1/2} (2\alpha)^{-1} e^{-v_2 t} e^{-x^2/(4d_2 t)} \\ &\quad - [\rho_2(D_1 - d_1) - \rho_1 E_2](4\pi t d_1)^{-1/2} (2\alpha)^{-1} e^{-v_1 t} e^{-x^2/(4d_1 t)} + O(t^{1/2}). \end{aligned} \tag{6.9}$$

It is interesting to note that

$$k = v_1 v_2 - \varepsilon_1 \varepsilon_2, \tag{6.10}$$

where v_i, ε_i $(i = 1, 2)$, are constants defined by (3.5) and (6.3), respectively. From (6.9) we see that for small times $u_1(x,t)$ and $u_2(x,t)$ involve the classical expressions with diffusivities d_1 and d_2, respectively. Since $d_1 + d_2 = D_1 + D_2$ we note that (1.9) implies that for the source solutions the large time diffusivity is less than the average of the small time diffusivities.

7. Solutions in Terms of Heat Functions

In this section we briefly indicate how solutions $\{u_1, u_2\}$ of (1.1) can be expressed in terms of solutions $\{h_1, h_2\}$ of the classical heat equation (1.10) as detailed in Theorem 3. We first consider solutions of (1.1) which vanish at infinity and are subject to initial conditions (3.2). Using the method described in Section 5 we can deduce expressions (3.3), where $\{h_1, h_2\}$ satisfy (1.10) and initial conditions (3.1), and $K_i(t, \xi)$ $(i = 1, 2, 3, 4)$ are given by (3.4) while the various constants and η are given by (3.5). In order to see that (3.3) are solutions of boundary value problems with prescribed initial conditions and zero on some boundary we need to verify that (3.3) are solutions of (1.1). This is done by directly substituting (3.3) into (1.1) and using integration by parts. The calculation is extremely long and tedious. Clearly, if $\{h_1, h_2\}$ satisfy the same initial conditions and are also zero on the same boundary, then (3.3) gives the appropriate solution of the boundary value problem. Thus solutions to a number of boundary value problems can now be deduced. A list of solutions for $\{h_1, h_2\}$ with zero boundary data can be found in Hill and Aifantis [9]. For problems with nonzero boundary data we can still make use of (3.3) except that we need to be able to transfer the boundary conditions on $\{u_1, u_2\}$ to boundary conditions on $\{h_1, h_2\}$. The final result is given in Theorem 4, the proof of which is outlined in Section 8.

8. The Inversion Formulae

In this section we find the inverse relations of (3.3). To obtain these relations we use essentially the Laplace transform method described in Hill and Aifantis [9]. Although the basic method is that given in [9] the details are considerably more complex. We first substitute for $K_i(t, \xi)$ $(i = 1, 2, 3, 4)$, from (3.4) into (3.3) and then change the integration variable. We let the new variable τ be defined by

$$\xi = d_2 t + (d_1 - d_2)\,\tau, \tag{8.1}$$

and assuming Dirichlet expansions of type (8.14) for $\{h_1, h_2\}$, we obtain expressions for $\{u_1, u_2\}$ involving integrals of the form

$$\int_0^t e^{-\lambda\tau} I_0(\zeta)\,d\tau, \qquad \int_0^t \frac{e^{-\lambda\tau} t I_1(\zeta)\,d\tau}{[\tau(t-\tau)]^{1/2}}, \qquad \int_0^t \frac{e^{-\lambda\tau}(t-2\tau)\,I_1(\zeta)\,d\tau}{[\tau(t-\tau)]^{1/2}}, \tag{8.2}$$

where λ denotes an arbitrary constant, ζ is defined by

$$\zeta = 2k^{1/2}[\tau(t-\tau)]^{1/2}, \tag{8.3}$$

and k is given by (3.5i). To obtain the Laplace transform of $\{u_1, u_2\}$ we need the following identities: For Re(s) sufficiently large and positive, λ real, and for ζ given by (8.3) we have (see, for example, [11]),

$$\int_0^\infty e^{-st}\int_0^t e^{-\lambda\tau} I_0(\zeta)\,d\tau\,dt = \frac{1}{(s^2+\lambda s-k)},$$

$$\int_0^\infty e^{-st}\int_0^t \frac{e^{-\lambda\tau} t I_1(\zeta)\,d\tau\,dt}{[\tau(t-\tau)]^{1/2}} = \frac{k^{1/2}}{s(s^2+\lambda s-k)} + \frac{k^{1/2}}{(s+\lambda)(s^2+\lambda s-k)},$$

$$\int_0^\infty e^{-st}\int_0^t \frac{e^{-\lambda\tau}(t-2\tau)\,I_1(\zeta)\,d\tau\,dt}{[\tau(t-\tau)]^{1/2}} = \frac{k^{1/2}}{s(s^2+\lambda s-k)} - \frac{k^{1/2}}{(s+\lambda)(s^2+\lambda s-k)}. \tag{8.4}$$

In taking the Laplace transform of $\{u_1, u_2\}$ we need to consider the Laplace transforms of quantities such as

$$z(t) = e^{(\delta-\gamma d_2)t}\int_0^t e^{-2\beta\tau} I_0(\zeta)\,h(d_2 t + 2\alpha\tau)\,d\tau, \tag{8.5}$$

where the space variable has been deleted for clarity. We assume that h can be expanded in a Dirichlet series so that

$$h(t) = \sum_{n=1}^{\infty} a_n e^{-k_n t}, \tag{8.6}$$

where a_n and k_n are independent of t and $k_n > 0$. Assuming that $\operatorname{Re}(k_n + s) > 0$ for all n, we have formally

$$\hat{h}(s) = \sum_{n=1}^{\infty} [a_n(k_n+s)^{-1}]. \tag{8.7}$$

Upon substituting (8.6) into (8.5) and taking the Laplace transform we obtain

$$\hat{z}(s) = \sum_{n=1}^{\infty} \{a_n[d_1 d_2(k_n+p)(k_n+q)]^{-1}\}, \tag{8.8}$$

where p and q are the roots of the quadratic equation in κ,

$$c\kappa^2 - (d+as)\,\kappa + (s^2+bs+e) = 0, \tag{8.9}$$

and where a–e are defined by (5.2). Written explicitly p and q are given by

$$p(s) = (2c)^{-1}\,\{(d+as) + [(d+as)^2 - 4c(s^2+bs+e)]^{1/2}\},$$

$$q(s) = (2c)^{-1}\,\{(d+as) - [(d+as)^2 - 4c(s^2+bs+e)]^{1/2}\}. \tag{8.10}$$

Using partial fractions and (8.7) we can express (8.8) in the form

$$\hat{z}(s)=[d_1d_2(p-q)]^{-1}\,[\hat{h}(q)-\hat{h}(p)]. \tag{8.11}$$

Using (8.4), in a similar manner we obtain the Laplace transforms,

$$\begin{aligned}
&\mathscr{L}\left\{e^{(\delta-\gamma d_2)t}\int_0^t \frac{e^{-2\beta\tau}I_1(\zeta)\,th(d_2t+2\alpha\tau)}{[\tau(t-\tau)]^{1/2}}\,d\tau\right\}\\
&\quad=\sum_{i=1}^{2}\frac{k^{1/2}[(q-p)\,\hat{h}(z_i)+(p-z_i)\,\hat{h}(q)+(z_i-q)\,\hat{h}(p)]}{cd_{3-i}[z_i^2(q-p)+q^2(p-z_i)+p^2(z_i-q)]},\\
&\mathscr{L}\left\{e^{(\delta-\gamma d_2)t}\int_0^t \frac{e^{-2\beta\tau}I_1(\zeta)(t-2\tau)\,h(d_2t+2\alpha\tau)}{[\tau(t-\tau)]^{1/2}}\,d\tau\right\}\\
&\quad=\sum_{i=1}^{2}\frac{(-1)^i\,k^{1/2}[(q-p)\,\hat{h}(z_i)+(p-z_i)\,\hat{h}(q)+(z_i-q)\,\hat{h}(p)]}{cd_{3-i}[z_i^2(q-p)+q^2(p-z_i)+p^2(z_i-q)]},
\end{aligned} \tag{8.12}$$

where z_i $(i=1,2)$, are defined by

$$z_1=d_2^{-1}(s+v_2), \qquad z_2=d_1^{-1}(s+v_1), \tag{8.13}$$

and where d_i, v_i $(i=1,2)$, are defined by (3.5). Assuming $\{h_1,h_2\}$ can be expressed as Dirichlet expansions, namely,

$$h_j(\mathbf{x},t)=\sum_{n=1}^{\infty}a_{nj}(\mathbf{x})\,e^{-k_{nj}t}, \qquad (j=1,2), \tag{8.14}$$

where $a_{nj}(\mathbf{x})$ $(j=1,2)$ are functions of position only and k_{nj} $(j=1,2)$, are positive constants, then the Laplace transform of $\{u_1,u_2\}$ can be deduced,

$$\begin{aligned}
\hat{u}_1(\mathbf{x},s)=\frac{k}{2\alpha c(p-q)}\Bigg\{&\left[\frac{m}{k}+\frac{(D_2-d_2)}{d_2(z_1-q)}-\frac{(D_2-d_1)}{d_1(z_2-q)}\right]\hat{h}_1(\mathbf{x},q)\\
&-\left[\frac{m}{k}+\frac{(D_2-d_2)}{d_2(z_1-p)}-\frac{(D_2-d_1)}{d_1(z_2-p)}\right]\hat{h}_1(\mathbf{x},p)\\
&+\left[\frac{2(\alpha B_1+\beta E_1)}{k}-\frac{E_1}{d_2(z_1-q)}+\frac{E_1}{d_1(z_2-q)}\right]\hat{h}_2(\mathbf{x},q)\\
&-\left[\frac{2(\alpha B_1+\beta E_1)}{k}-\frac{E_1}{d_2(z_1-p)}+\frac{E_1}{d_1(z_2-p)}\right]\hat{h}_2(\mathbf{x},p)\Bigg\},
\end{aligned}$$

$$\hat{u}_2(\mathbf{x}, s) = \frac{k}{2\alpha c(p-q)} \left\{ \left[\frac{m}{k} + \frac{(D_1 - d_1)}{d_1(z_2 - p)} - \frac{(D_1 - d_2)}{d_2(z_1 - p)}\right] \hat{h}_2(\mathbf{x}, p) \right.$$
$$- \left[\frac{m}{k} + \frac{(D_1 - d_1)}{d_1(z_2 - q)} - \frac{(D_2 - d_2)}{d_2(z_1 - q)}\right] \hat{h}_2(\mathbf{x}, q)$$
$$+ \left[\frac{2(\alpha B_2 + \beta E_2)}{k} - \frac{E_2}{d_2(z_1 - q)} + \frac{E_2}{d_1(z_2 - q)}\right] \hat{h}_1(\mathbf{x}, q)$$
$$\left. - \left[\frac{2(\alpha B_2 + \beta E_2)}{k} - \frac{E_2}{d_2(z_1 - p)} + \frac{E_2}{d_1(z_2 - p)}\right] \hat{h}_1(\mathbf{x}, p) \right\}, \tag{8.15}$$

where m is defined by

$$m = \beta(D_1 - D_2) - \alpha(A_1 - A_2). \tag{8.16}$$

To find $\{\hat{h}_1, \hat{h}_2\}$ in terms of $\{\hat{u}_1, \hat{u}_2\}$ we assume a general form for $\{\hat{h}_1, \hat{h}_2\}$ and substitute it into (8.15). This leads to a system of twelve equations in twelve unknowns. Once this system is solved we have determined $\{\hat{h}_1, \hat{h}_2\}$ in terms of $\{\hat{u}_1, \hat{u}_2\}$. Let $s_+(s)$ and $s_-(s)$ be the roots of (8.9) and we adopt the following notation:

$$P = P(s) = s_+(s), \qquad Q = Q(s) = s_-(s),$$
$$s_+(q) = 2^{-1}\{(qa - b) + [(qa - b)^2 - 4(cq^2 - dq + e)]^{1/2}\},$$
$$s_-(q) = 2^{-1}\{(qa - b) - [(qa - b)^2 - 4(cq^2 - dq + e)]^{1/2}\}, \tag{8.17}$$

where $a - e$ are defined by (5.2). We assume $\{\hat{h}_1, \hat{h}_2\}$ to be of the following form:

$$\hat{h}_1(\mathbf{x}, s) = \frac{1}{(P-Q)} \left\{ \left[\theta_1 + \frac{\theta_2}{(Z_1 - Q)} + \frac{\theta_3}{(Z_2 - Q)}\right] \hat{u}_1(\mathbf{x}, Q) \right.$$
$$- \left[\theta_1 + \frac{\theta_2}{(Z_1 - P)} + \frac{\theta_3}{(Z_2 - P)}\right] \hat{u}_1(\mathbf{x}, P)$$
$$+ \left[\theta_4 + \frac{\theta_5}{(Z_1 - Q)} + \frac{\theta_6}{(Z_2 - Q)}\right] \hat{u}_2(\mathbf{x}, Q)$$
$$\left. - \left[\theta_4 + \frac{\theta_5}{(Z_1 - P)} + \frac{\theta_6}{(Z_2 - P)}\right] \hat{u}_2(\mathbf{x}, P) \right\},$$

$$\hat{h}_2(\mathbf{x}, s) = \frac{1}{(P-Q)} \left\{ \left[\theta_7 + \frac{\theta_8}{(Z_1 - Q)} + \frac{\theta_9}{(Z_2 - Q)}\right] \hat{u}_2(\mathbf{x}, Q) \right.$$
$$- \left[\theta_7 + \frac{\theta_8}{(Z_1 - P)} + \frac{\theta_9}{(Z_2 - P)}\right] \hat{u}_2(\mathbf{x}, P)$$

$$+\left[\theta_{10}+\frac{\theta_{11}}{(Z_1-Q)}+\frac{\theta_{12}}{(Z_2-Q)}\right]\hat{u}_1(\mathbf{x}, Q)$$

$$-\left[\theta_{10}+\frac{\theta_{11}}{(Z_1-P)}+\frac{\theta_{12}}{(Z_2-P)}\right]\hat{u}_1(\mathbf{x}, P)\Bigg\}, \tag{8.18}$$

where θ_j ($j=1, 2,..., 12$), are constants to be determined and Z_i ($i=1, 2$), are given by

$$Z_1 = d_2 s - v_2, \qquad Z_2 = d_1 s - v_1. \tag{8.19}$$

To substitute (8.18) into (8.15) we need to evalute P and Q at the values $s=p$ and q. We obtain

$$P(p)=pa-b-s, \qquad P(q)=s, \qquad Q(p)=s,$$
$$Q(q)=qa-b-s. \tag{8.20}$$

The following identities are also required:

$$(P-Z_1)(Q-Z_1)=-k=(P-Z_2)(Q-Z_2),$$
$$(P-Z_1)(P-Z_2)=k=(Q-Z_1)(Q-Z_2). \tag{8.21}$$

Upon substitution of (8.18) into (8.15) and collecting like terms we obtain two similar systems of six equations in six unknowns which are identities in the independent variable s. For example, from the system involving θ_1, θ_2, θ_3, θ_{10}, θ_{11}, and θ_{12} we can (by considering the coefficients of s^0, s, s^2, $[(d+as)^2-4c(s^2+bs+e)]^{1/2}$, and $s[(d+as)^2-4c(s^2+bs+e)]^{1/2}$) extract the following system of four equations in four unknowns

$$d_2^2(d_1-D_2)\,\theta_2+d_1^2(d_2-D_2)\,\theta_3+d_2^2E_1\theta_{11}+d_1^2E_1\theta_{12}=0,$$
$$d_2^2(d_1-D_2)\,\theta_2-d_1^2(d_2-D_2)\,\theta_3+d_2^2E_1\theta_{11}-d_1^2E_1\theta_{12}=0,$$
$$d_1E_2\theta_2-d_2E_2\theta_3+d_1(d_2-D_1)\,\theta_{11}-d_2(d_1-D_1)\,\theta_{12}=0,$$
$$(D_2+E_2)(\theta_2+\theta_3)-(D_1+E_1)(\theta_{11}+\theta_{12})=ck. \tag{8.22}$$

For further details we refer the reader to Lee [10]. The other system of six equations involving θ_4, θ_5, θ_6, θ_7, θ_8, and θ_9 is treated in the same manner. The final expressions for θ_i ($i=1, 2,..., 12$), are given by

$$\theta_1=B_2E_1+(4a^2)^{-1}\,[aE_1E_2(A_1-A_2)+(2c-aD_2)(B_1E_2+B_2E_1)],$$
$$\theta_2=(2a)^{-1}\,kd_2(d_1-D_1), \qquad \theta_3=-(2a)^{-1}\,kd_1(d_2-D_1),$$
$$\theta_4=-B_1D_2-(4a^2)^{-1}\,E_1[(A_1-A_2)(2c-aD_2)-a(B_1E_2+B_2E_1)],$$

$$\theta_5 = -(2\alpha)^{-1} kd_2E_1, \qquad \theta_6 = (2\alpha)^{-1} kd_1E_1,$$

$$\theta_7 = B_1E_2 - (4\alpha^2)^{-1} [aE_1E_2(A_1 - A_2) - (2c - aD_1)(B_1E_2 + B_2E_1)],$$

$$\theta_8 = (2\alpha)^{-1} kd_2(d_1 - D_2), \qquad \theta_9 = -(2\alpha)^{-1} kd_1(d_2 - D_2),$$

$$\theta_{10} = -B_2D_1 + (4\alpha^2)^{-1} E_2[(A_1 - A_2)(2c - aD_1) + a(B_1E_2 + B_2E_1)],$$

$$\theta_{11} = -(2\alpha)^{-1} kd_2E_2, \qquad \theta_{12} = (2\alpha)^{-1} kd_1E_2, \tag{8.23}$$

where a and c are defined in (5.2). The inverse Laplace transforms of (8.18) can be obtained by using the method described in Eqs. (8.5)–(8.11). For example, the term involving θ_1 gives

$$\mathscr{L}\left\{c^{-1}e^{v_1t/d_1}\int_0^t e^{-2\alpha\delta\tau/c}I_0(\zeta^*)\,u_1(\mathbf{x}, d_1^{-1}t + 2ac^{-1}\tau)\,d\tau\right\}$$
$$= (P - Q)^{-1} [\hat{u}_1(\mathbf{x}, Q) - \hat{u}_1(\mathbf{x}, P)], \tag{8.24}$$

where ζ^* is defined by

$$\zeta^* = 2k^{1/2}c^{-1/2}[\tau(t - \tau)]^{1/2}. \tag{8.25}$$

Similar formulae can be obtained for the other terms involving θ_i ($i = 2, 3, \ldots, 12$). Letting

$$\tau = (2\alpha)^{-1} c(\xi - d_1^{-1}t), \tag{8.26}$$

then from (8.18) and equations similar to (8.24) we obtain (3.6)–(3.8).

9. Standard Boundary Data

To illustrate the formulae derived in Section 8, we deduce boundary conditions on $\{h_1, h_2\}$ for given standard boundary conditions on $\{u_1, u_2\}$. We obtain results for the following boundary data:

$$u_1(\mathbf{x}, t) = A, \qquad u_2(\mathbf{x}, t) = B, \qquad \mathbf{x} \in \partial\Omega, \tag{9.1a}$$

$$u_1(\mathbf{x}, t) = Ae^{-lt}, \qquad u_2(\mathbf{x}, t) = Be^{-lt}, \qquad \mathbf{x} \in \partial\Omega, \tag{9.1b}$$

where A, B, and l denote arbitrary constants and $\partial\Omega$ denotes the boundary containing the domain of $\mathbf{x}$. The boundary conditions on $\{h_1, h_2\}$ arising from (9.1a) can be obtained from those arising from (9.1b) by setting $l = 0$. Consequently, we describe the derivation of the conditions arising from (9.1b) only. In (3.6) we make the change of variable (8.26). We then have to evaluate integrals similar to (8.5). Using (8.4) we obtain the following formulae:

$$e^{\nu_1 t/d_1} \int_0^t e^{-2\alpha\delta\tau/c} e^{-l(t/d_1+2\alpha\tau/c)} I_0(\zeta^*)\, d\tau$$

$$= -\frac{(e^{-\lambda_1 t} - e^{-\lambda_2 t})}{(\lambda_1 - \lambda_2)},$$

$$e^{\nu_1 t/d_1} \int_0^t e^{-2\alpha\delta\tau/c} e^{-l(t/d_1+2\alpha\tau/c)} I_1(\zeta^*) \left(\frac{\tau}{t-\tau}\right)^{1/2} d\tau$$

$$= \left(\frac{k}{c}\right)^{1/2} \left[\frac{e^{-\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{e^{-\lambda_2 t}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} + \frac{e^{-\lambda_3 t}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right],$$

$$e^{\nu_1 t/d_1} \int_0^t e^{-2\alpha\delta\tau/c} e^{-l(t/d_1+2\alpha\tau/c)} I_1(\zeta^*) \left(\frac{t-\tau}{\tau}\right)^{1/2} d\tau$$

$$= \left(\frac{k}{c}\right)^{1/2} \left[\frac{e^{-\lambda_1 t}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)} - \frac{e^{-\lambda_2 t}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_4)} + \frac{e^{-\lambda_4 t}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)}\right], \tag{9.2}$$

where ζ^* is defined by (8.25) and λ_1, λ_2, λ_3, and λ_4 are given by

$$\lambda_1 = (2c)^{-1}\{(al-d) + [(al-d)^2 - 4c(l^2 - bl + e)]^{1/2}\},$$
$$\lambda_2 = (2c)^{-1}\{(al-d) - [(al-d)^2 - 4c(l^2 - bl + e)]^{1/2}\},$$
$$\lambda_3 = d_2^{-1}(l - \nu_2), \qquad \lambda_4 = d_1^{-1}(l - \nu_1). \tag{9.3}$$

The resulting equations can be simplified by use of the following consequences of (9.3):

$$(\lambda_1-\lambda_3)(\lambda_2-\lambda_3) = -kc^{-1} = (\lambda_1-\lambda_4)(\lambda_2-\lambda_4),$$
$$(\lambda_1-\lambda_3)(\lambda_1-\lambda_4) = kc^{-1} \quad = (\lambda_2-\lambda_3)(\lambda_2-\lambda_4). \tag{9.4}$$

The final expressions for the boundary conditions on $\{h_1, h_2\}$ arising from (9.1b) are given by

$$h_1(\mathbf{x}, t) = (\lambda_1-\lambda_2)^{-1}\{A(\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}) + c^{-1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$
$$\times [A(A_2 D_1 + B_1 E_2 - D_1 l) + B(A_2 E_1 + B_1 D_2 - E_1 l)]\},$$
$$h_2(\mathbf{x}, t) = (\lambda_1-\lambda_2)^{-1}\{B(\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}) + c^{-1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$
$$\times [B(A_1 D_2 + B_2 E_1 - D_2 l) + A(A_1 E_2 + B_2 D_1 - E_2 l)]\}, \tag{9.5}$$

where λ_i ($i = 1, 2, 3, 4$), are given by (9.3) and where we have used the identities

$$D_1\delta - c\gamma + \theta_1 = -(A_2 D_1 + B_1 E_2), \qquad E_1\delta + \theta_4 = -(A_2 E_1 + B_1 D_2),$$
$$D_2\delta - c\gamma + \theta_7 = -(A_1 D_2 + B_2 E_1), \qquad E_2\delta + \theta_{10} = -(A_1 E_2 + B_2 D_1). \tag{9.6}$$

For the boundary data (9.1a) we put $l=0$ in (9.5) to obtain

$$\begin{aligned} h_1(\mathbf{x}, t) = {} & (4c\varepsilon)^{-1}\, e^{dt/(2c)}\{A[e^{\varepsilon t}(d+2c\varepsilon) - e^{-\varepsilon t}(d-2c\varepsilon)] \\ & - 2(e^{\varepsilon t} - e^{-\varepsilon t})[A(A_2 D_1 + B_1 E_2) + B(A_2 E_1 + B_1 D_2)]\}, \\ h_2(\mathbf{x}, t) = {} & (4c\varepsilon)^{-1}\, e^{dt/(2c)}\{B[e^{\varepsilon t}(d+2c\varepsilon) - e^{-\varepsilon t}(d-2c\varepsilon)] \\ & - 2(e^{\varepsilon t} - e^{-\varepsilon t})[B(A_1 D_2 + B_2 E_1) + A(A_1 E_2 + B_2 D_1)]\}, \end{aligned} \tag{9.7}$$

where the various constants are as previously defined and ε is defined by

$$\varepsilon = (2c)^{-1}\,(d^2 - 4ce)^{1/2}. \tag{9.8}$$

We note that we can obtain boundary expressions for $\{h_1, h_2\}$ arising from boundary data of the type

$$u_1(\mathbf{x}, t) = At^n, \qquad u_2(\mathbf{x}, t) = Bt^n, \qquad \mathbf{x} \in \partial\Omega, \tag{9.9}$$

where n is a positive integer, by differentiating the right-hand side of (9.5) n times with respect to l, multiplying by $(-1)^n$, and putting $l=0$. We also note that the results given in this section can be utilized for boundary data of the type

$$\frac{\partial u_1}{\partial n}(\mathbf{x}, t) = A, \qquad \frac{\partial u_2}{\partial n}(\mathbf{x}, t) = B, \qquad \mathbf{x} \in \partial\Omega, \tag{9.10a}$$

$$\frac{\partial u_1}{\partial n}(\mathbf{x}, t) = Ae^{-lt}, \qquad \frac{\partial u_2}{\partial n}(\mathbf{x}, t) = Be^{-lt}, \qquad \mathbf{x} \in \partial\Omega. \tag{9.10b}$$

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